Multi-Dof systems with damping

$M\ddot{\vec{x}} + K\vec{x} = B_f f(t)$ ⟶ physical coordinates

$\vec{X} = M^{-1/2}\vec{q}$ ⟶ pre multiply by $M^{-1/2}$

$$M^{-1/2} M M^{-1/2} \ddot{\vec{q}} + M^{-1/2} K M^{-1/2} \vec{q} = M^{-1/2} B_f f(t)$$

$= 0$ (homogeneous e-values/e-vectors)

$$I\ddot{q} + \underset{M^{-1}K}{\tilde{K}}\vec{q} = 0$$

$M^{-1/2} K M^{-1/2}$ ⟶ symmetric, positive definite, yields orthogonal eigenvectors

$M^{-1}K$ ⟶ non-symmetric (symmetric if $m_1 = m_2$)

$|\tilde{K} - \lambda I| = 0$, λ_1, λ_2

e-vectors V_1, V_2 are orthogonal

Create a modal matrix $P = [V_1, V_2]$ Then $P^T P = I$, where V_i's are orthonormal

$V_1^T V_1 = 1$

$V_1^T V_2 = 0$

$M^{-1}K$: u_1, u_2

-problem: $u_1^T u_2 \neq 0$

spanning the space means $x(t) = 2\bigcirc u_1 + \bigcirc u_2$ (fill in, fill in)

Coordinate transformation

$\vec{X} = M^{-1/2}\vec{q}$ $\quad\quad \vec{q} = M^{1/2}\vec{X}$

Implies: $I\ddot{q} + \tilde{K}\vec{q} = M^{-1/2} B_f f(t)$

Another transformation: $\vec{q} = P\vec{r}$

$$I P \ddot{\vec r} + \tilde K P \vec r = M^{-1/2} B_f f(t)$$

premultiply by P^T

$$P^T I P \ddot{\vec r} + P^T \tilde K P \vec r = P^T M^{-1/2} B_f f(t)$$

$$I \ddot r + \Lambda \vec r = P^T M^{-1/2} B_f f(t)$$

$\vec r$ = modal coordinates

$$\vec r = \begin{bmatrix} r_1(t) \\ r_2(t) \\ \cdots \end{bmatrix}$$

Solution can be written as a linear combination of mode shapes

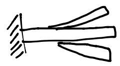
Projecting now from physical to modal

Example: $m_1 = 9$, $m_2 = 1$, $K_1 = 24$, $K_2 = 3$ > decouple the system using $M^{-1}K$, then symmetric eigenvalue approach

$\omega_1^2 = 2$, $\omega_2^2 = 4$

$$I \ddot{\vec r} + \Lambda_K \vec r = \Gamma_i f(t)$$

$\Gamma(i)$ = modal participation factors

$\lambda = 1$  1 mode

$\lambda = 2$ 2 modes, $\lambda = 3$ 3 modes

<u>Damping</u>: in MDof Systems

$$M \ddot{\vec X} + D \dot{\vec X} + K \vec X = B_f f(t)$$

$$D = \alpha M + \beta K$$

α, β are chosen and allow a form of damping that permits us to decouple the systems

$\vec X = M^{-1/2} \vec q$ $\qquad$ $\vec q = P \vec r$

$$\boxed{\vec X = M^{-1/2} P \vec r}$$

$$M M^{-1/2} P \ddot{\vec{r}} + D M^{-1/2} P \dot{\vec{r}} + K M^{-1/2} P \vec{r} = B_f f(t)$$

Assuming we know P from $\tilde{K}$

- premultiply by $P^T M^{-1/2}$

$$P^T M^{-1/2} M M^{-1/2} P \ddot{\vec{r}} + P^T M^{-1/2} D M^{-1/2} P \dot{\vec{r}} + P^T M^{-1/2} K M^{-1/2} P \vec{r} = P^T M^{-1/2} B_f f(t)$$

$$I \ddot{\vec{r}} + \Lambda_D \dot{\vec{r}} + \Lambda_K \vec{r} = \vec{\Gamma} f(t) \qquad \vec{\Gamma} = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \end{bmatrix}$$

$$\Lambda_D = P^T M^{-1/2} D M^{-1/2} P = P^T M^{-1/2} (\alpha M + \beta K) M^{-1/2} P$$

$$= \alpha P^T M^{-1/2} M M^{-1/2} P + \beta P^T M^{-1/2} K M^{-1/2} P$$

$$\Lambda_D = \alpha I + \beta \Lambda_K = \mathrm{diag}\left[2 \zeta_i \omega_i\right]$$

normal modes: - nodes are fixed

- modes being real numbers not complex

$$\vec{U} = M^{1/2} \vec{V}_i$$

$$\vec{X} = M^{-1/2} \vec{q}_i$$

$M^{-1}K$ always gives mode shapes, but they might not decouple the system